# The Magnetic Field

Now that we have defined **magnetization current**, we find that Ampere's Law for fields **within some material** becomes:

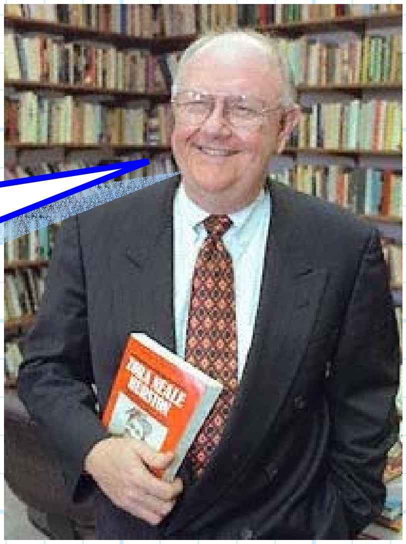
$$\begin{aligned}\nabla\times\mathbf{B}(\bar{r}) &= \mu_0\left(\mathbf{J}(\bar{r})+\mathbf{J}_m(\bar{r})\right)\\ &= \mu_0\left(\mathbf{J}(\bar{r})+\nabla\times\mathbf{M}(\bar{r})\right)\end{aligned}$$

This of course is **analogous** to the expression we derived for **Gauss's Law** in a dielectric media:

$$\nabla\cdot\mathbf{E}(\bar{r})=\frac{\rho_v(\bar{r})+\rho_{vp}(\bar{r})}{\varepsilon_0}=\frac{\rho_v(\bar{r})-\nabla\cdot\mathbf{P}(\bar{r})}{\varepsilon_0}$$

Recall that we **removed** the polarization charge from this expression by defining a **new** vector field $\mathbf{D}(\bar{r})$, leaving us with the more **general** expression of Gauss's Law:

$$\nabla\cdot\mathbf{D}(\bar{r})=\rho_v(\bar{r})$$



**Q:** *Can we similarly define a* ***new*** *vector field to "take care" of* ***magnetization*** *current ??*

**A:** Yes! We call this vector field the **magnetic field** $\mathbf{H}(\bar{r})$.

Let's begin by **rewriting** Ampere's Law as:

$$\nabla \times \mathbf{B}(\bar{r}) - \mu_0 \mathbf{J}_m(\bar{r}) = \mu_0 \mathbf{J}(\bar{r})$$

Yuck! Now we see clearly the problem. In **free space**, if we know current distribution $\mathbf{J}(\bar{r})$, we can find the resulting magnetic flux density $\mathbf{B}(\bar{r})$ using the **Biot-Savart** Law:

$$\mathbf{B}(\bar{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}(\bar{r}') \times (\bar{r} - \bar{r}')}{|\bar{r} - \bar{r}'|^3}\, dv'$$

But this is the solution for current in **free space**! It is **no longer valid** if some **material** is present!

**Q:** *Why?*

**A:** Because, the magnetic flux density produced by current $\mathbf{J}(\bar{r})$ may **magnetize** the material (i.e., produce magnetic dipoles), thus producing **magnetization currents** $\mathbf{J}_m(\bar{r})$ .

These magnetization currents $\mathbf{J}_m(\bar{r})$ will **also** produce a magnetic flux density—a **modification** of vector field $\mathbf{B}(\bar{r})$ that is **not** accounted for in the Biot-Savart expression shown above!

To determine the correct solution, we first recall that:

$$\mathbf{J}_m(\bar{r}) = \nabla \times \mathbf{M}(\bar{r})$$

Therefore Ampere's Law is:

$$\nabla \times \mathbf{B}(\bar{r}) - \mu_0 \nabla \times \mathbf{M}(\bar{r}) = \mu_0 \mathbf{J}(\bar{r})$$

$$\nabla \times \left[\mathbf{B}(\bar{r}) - \mu_0 \mathbf{M}(\bar{r})\right] = \mu_0 \mathbf{J}(\bar{r})$$

$$\nabla \times \left[\frac{\mathbf{B}(\bar{r})}{\mu_0} - \mathbf{M}(\bar{r})\right] = \mathbf{J}(\bar{r})$$

Now let's define a **new** vector field $\mathbf{H}(\bar{r})$, called the **magnetic field**:

$$\mathbf{H}(\bar{r}) \doteq \frac{\mathbf{B}(\bar{r})}{\mu_0} - \mathbf{M}(\bar{r}) \qquad \left[\frac{Amps}{meter}\right]$$

**Ampere's Law** therefore can be written in terms of the **magnetic field** as:

$$\nabla \times \mathbf{H}(\bar{r}) = \mathbf{J}(\bar{r})$$

Hey! We **know** what the solution to **this** differential equation is! Recall the solution to:

$$\nabla \times \mathbf{B}(\bar{r}) = \mu_0 \mathbf{J}(\bar{r})$$

is the **Biot-Savart Law**.

If we make the **substitution**:

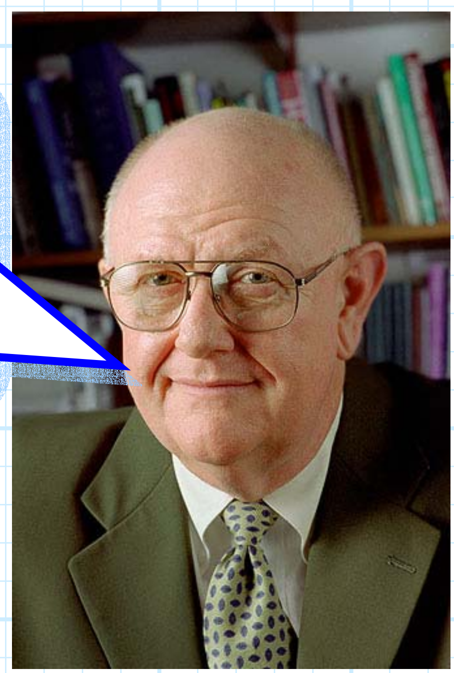
$$\mathbf{H}(\bar{r}) \leftrightarrow \frac{\mathbf{B}(\bar{r})}{\mu_0}$$

we find that both differential **equations** are identical. Therefore their **solutions** are also identical when making the **same** substitution.

Making this substitution into the Biot-Sarvart Law, we find that:

$$\mathbf{H}(\bar{\mathrm{r}}) = \frac{1}{4\pi} \iiint_V \frac{\mathbf{J}(\bar{\mathrm{r}}') \times (\bar{\mathrm{r}} - \bar{\mathrm{r}}')}{\left|\bar{\mathrm{r}} - \bar{\mathrm{r}}'\right|^3} \, dv'$$

**Q:** *Swell. But may I remind you that we were* ***suppose*** *to be finding the solution for the &%^@!+*#&* ***magnetic flux density*** $\mathrm{B}(\bar{r})$!



True! But since we can find $\mathbf{H}(\bar{r})$ from $\mathbf{J}(\bar{r})$, our task **now** is to determine the **relationship** between $\mathbf{B}(\bar{r})$ and $\mathbf{H}(\bar{r})$.

We call the relationship between $\mathbf{B}(\bar{r})$ and $\mathbf{H}(\bar{r})$ a **constitutive equation**. For most media, we find that the magnetization vector $\mathbf{M}(\bar{r})$ is directly **proportional** to the magnetic field $\mathbf{H}(\bar{r})$:

$$\mathbf{M}(\bar{r}) = \chi_m \, \mathbf{H}(\bar{r})$$

where the proportionality coefficient $\chi_m$ is the **magnetic susceptibility** of the material.

* Note that for a given magnetic field $\mathbf{H}(\bar{r})$, as $\chi_m$ **increases**, the magnetization vector $\mathbf{M}(\bar{r})$ **increases**.

* Magnetic susceptibility $\chi_m$ therefore indicates how **susceptible** the material is to **magnetization**.

* In other words, $\chi_m$ is a measure of how easily (or difficult) it is to create and align **magnetic dipoles** (from atoms/molecules) within the **material**.

Again, note the **analogy** to electrostatics. We defined earlier **electric** susceptibility $\chi_e$, which indicates how susceptible a material is to **polarization** (i.e., the creation of **electric** dipoles).

We can now determine the relationship between $\mathbf{B}(\bar{r})$ and $\mathbf{H}(\bar{r})$. Using the above expression, we find:

$$\mathbf{H}(\bar{r}) = \frac{\mathbf{B}(\bar{r})}{\mu_0} - \mathbf{M}(\bar{r})$$

$$\mathbf{H}(\bar{r}) = \frac{\mathbf{B}(\bar{r})}{\mu_0} - \chi_m \mathbf{H}(\bar{r})$$

$$\mathbf{H}(\bar{r}) + \chi_m \mathbf{H}(\bar{r}) = \frac{\mathbf{B}(\bar{r})}{\mu_0}$$

$$\mu_0 (1 + \chi_m) \mathbf{H}(\bar{r}) = \mathbf{B}(\bar{r})$$

Hey! Magnetic field $\mathbf{H}(\bar{r})$ and magnetic flux density are related by a **simple constant**!

$$\mathbf{B}(\bar{r}) = \mu \, \mathbf{H}(\bar{r})$$

where:

$$\mu \doteq \text{ material } \textbf{permeability} \qquad \left[ \frac{N}{A^2} = \frac{Henries}{m} \right]$$

$$= \mu_0 (1 + \chi_m)$$

We typically **further** simplify this expression by defining a **relative** permeability:

$$\mu_r \doteq \text{relative permeability}$$
$$= 1 + \chi_m$$

So that:

$$\mathbf{B}(\bar{r}) = \mu \mathbf{H}(\bar{r}) = \mu_0 \mu_r \mathbf{H}(\bar{r})$$

In other words, if the **relative** permeability of some material was, say, $\mu_r = 2$, then the **permeability** of the material is **twice** that of the permeability of **free space** (i.e., $\mu = 2\mu_0$). This perhaps is more readily evident when we write:

$$\mu_r = \frac{\mu}{\mu_0}$$

Note that $\mu$ and/or $\mu_r$ are **proportional** to magnetic susceptibility $\chi_m$. As a result, permeability is likewise an indication of how **susceptible** a material to **magnetization**.

* If $\mu_r = 1$, this susceptibility is that of **free space** (i.e., **none**!).

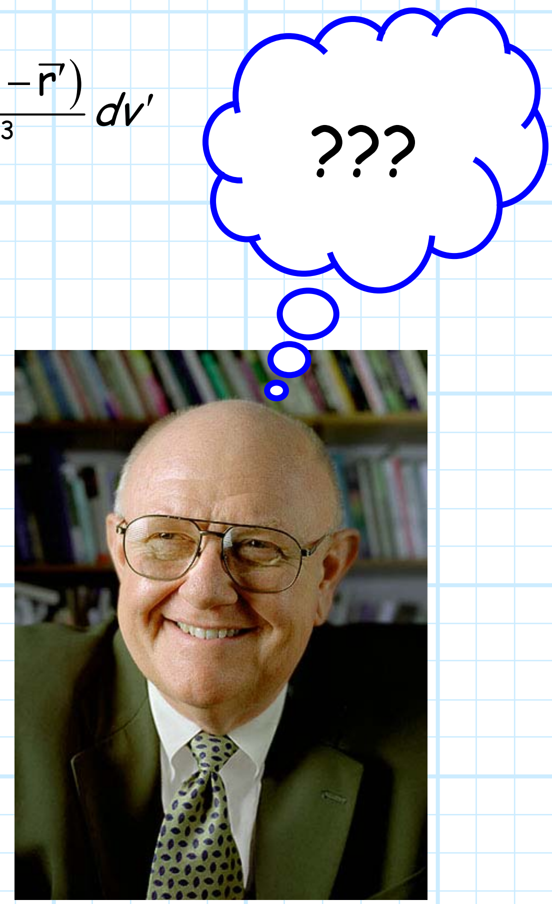
* Alternatively, a **large** $\mu_r$ indicates a material that is **easily magnetized**.

* For example, the relative permeability of **iron** is $\mu_r$=4000 !

**Now**, we are **finally** able to determine the **magnetic flux density** in some **material,** produced by current density $\mathbf{J}(\bar{r})$!

Since $\mathbf{B}(\bar{r}) = \mu \mathbf{H}(\bar{r})$ and:

$$\mathbf{H}(\bar{\mathrm{r}}) = \frac{1}{4\pi} \iiint_V \frac{\mathbf{J}(\bar{\mathrm{r}}') \times (\bar{\mathrm{r}} - \bar{\mathrm{r}}')}{|\bar{\mathrm{r}} - \bar{\mathrm{r}}'|^3} dv'$$




we find the desired solution:

$$\mathbf{B}(\bar{r}) = \frac{\mu}{4\pi} \iiint_V \frac{\mathbf{J}(\bar{\mathrm{r}}') \times (\bar{\mathrm{r}} - \bar{\mathrm{r}}')}{|\bar{\mathrm{r}} - \bar{\mathrm{r}}'|^3} dv'$$

**Comparing** this result with the Biot-Sarvart Law for **free space**, we see that the only difference is that $\mu_0$ has been replaced with $\mu$!

This last result is therefore is a **more general** form of the Biot-Savart Law, giving the correct result for fields within some **material** with permeability $\mu$. Of course, the "material" **could** be free space. However, the expression above will **still** provide the **correct** answer; because for free space $\mu = \mu_0$, thus returning the equation to its **original** (i.e., free space) form!

Summarizing, we can attribute the existence of a **magnetic field** $\mathbf{H}(\bar{r})$ to **conduction** current $\mathbf{J}(\bar{r})$, while we attribute the existence of **magnetic flux density** to the **total** current density, including the magnetization current.

$$\mathbf{J}(\bar{r}) \Rightarrow \mathbf{H}(\bar{r})$$

$$\mathbf{J}(\bar{r}) + \mathbf{J}_m(\bar{r}) \Rightarrow \mathbf{B}(\bar{r})$$

Finally, we again want to note the **analogies** between electrostatics and the magnetostatic expressions derived in this handout:

$$\mathbf{B}(\bar{r}) = \mu_0 \mathbf{H}(\bar{r}) + \mu_0 \mathbf{M}(\bar{r}) \quad \Leftrightarrow \quad \mathbf{D}(\bar{r}) = \varepsilon_0 \mathbf{E}(\bar{r}) + \mathbf{P}(\bar{r})$$

$$\mathbf{B}(\bar{r}) = \mu_0 (1 + \chi_m) \mathbf{H}(\bar{r}) \quad \Leftrightarrow \quad \mathbf{D}(\bar{r}) = \varepsilon_0 (1 + \chi_e) \mathbf{E}(\bar{r})$$

$$\mathbf{B}(\bar{r}) = \mu \mathbf{H}(\bar{r}) \quad \Leftrightarrow \quad \mathbf{D}(\bar{r}) = \varepsilon \mathbf{E}(\bar{r})$$

$$\mathbf{B}(\bar{r}) \Leftrightarrow \mathbf{D}(\bar{r})$$

$$\mathbf{H}(\bar{r}) \Leftrightarrow \mathbf{E}(\bar{r})$$

$$\mathbf{M}(\bar{r}) \Leftrightarrow \mathbf{P}(\bar{r})$$

$$\chi_m \Leftrightarrow \chi_e$$

$$\mu \Leftrightarrow \varepsilon$$